Oxford Cambridge and RSA

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension
Insert

Friday 15 June 2018 – Afternoon
Time allowed: 2 hours

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INFORMATION FOR CANDIDATES
- This Insert contains the article for Section B.
- This document consists of **4** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER/INVIGILATOR
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Arithmetic and Geometric Means

Arithmetic and geometric mean of two numbers

For two real numbers a and b, the arithmetic mean of the numbers is defined to be $\frac{a+b}{2}$. For two non-negative real numbers a and b, the geometric mean of the two numbers is defined to be $\sqrt{ab}$.

Squares of real numbers cannot be negative, so we know that $(a-b)^2 \geqslant 0$. It follows that $a^2+b^2 \geqslant 2ab$ and so $(a+b)^2 \geqslant 4ab$. Hence the arithmetic mean of two real non-negative numbers is greater than, or equal to, their geometric mean.

$$\frac{a+b}{2} \geqslant \sqrt{ab} \text{ for } a, b \geqslant 0$$

This result is known as the inequality of the arithmetic and geometric means. If the two numbers a and b are equal then the arithmetic mean equals the geometric mean.

The three real numbers a, $\frac{a+b}{2}$, b form an arithmetic sequence. The three non-negative real numbers a, $\sqrt{ab}$, b form a geometric sequence.

Constructing the arithmetic and geometric mean of two numbers

Lengths representing the arithmetic and geometric mean of two positive numbers can be constructed with a straight edge and compasses.

Fig. C1.1 shows a straight line ACB with AC of length a and CB of length b.

Fig. C1.1

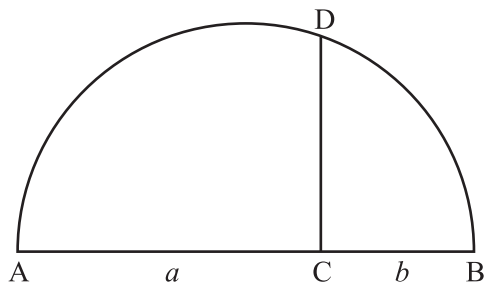

Fig. C1.2

The line AB is first bisected, to locate its midpoint. A semicircle with AB as diameter is then drawn, and a line at C perpendicular to the diameter is constructed. Fig. C1.2 shows this semicircle, with the perpendicular line through C meeting the semicircle at D.

The radius of the semicircle is the arithmetic mean of a and b, and the length of CD is the geometric mean of a and b.

To prove that the length of CD is the geometric mean of a and b, consider triangles ACD and BCD, as shown in Fig. C1.3. Letting angle CBD = θ, it follows that angle CDA is also θ. Finding expressions for $\tan\theta$ in each of triangles ACD and BCD leads to $h = \sqrt{ab}$, where h is the length of CD.

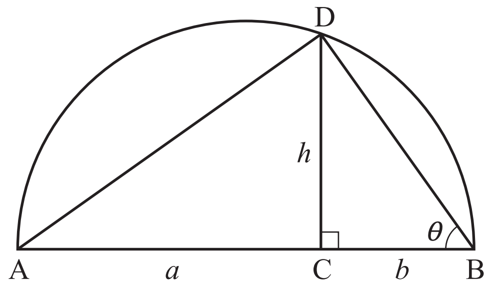

Fig. C1.3

The relationship between a, b and h in Fig. C1.3 means that a square with side CD has the same area as a rectangle with sides equal to AC and CB. Fig. C2 shows the square and a rectangle ACFG with CF equal in length to CB. This diagram illustrates how a straight edge and compasses can be used to construct a square with area equal to that of a given rectangle. This method appears in Euclid's books on Geometry (the 'Elements') which were published around 2300 years ago.

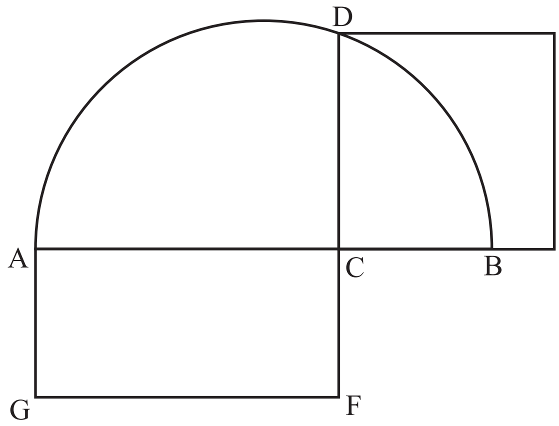

Fig. C2

Areas of rectangles

The inequality of arithmetic and geometric means implies that the square has the smallest perimeter of all rectangles with the same area.

Consider a rectangle of given area A that has sides of lengths x and y, so that $xy = A$. The perimeter of this rectangle is $2(x + y)$. From the inequality of arithmetic and geometric means, we know that $\frac{x+y}{2} \geqslant \sqrt{xy}$ so that $2(x+y) \geqslant 4\sqrt{xy}$. But the right-hand side of this last inequality has the fixed value $4\sqrt{A}$ whatever x and y are. For a square of area A, each side has length $\sqrt{A}$ and so $4\sqrt{A}$ is the perimeter of this square. Therefore, the perimeter of any rectangle of area A is not less than this, so the square has the smallest perimeter of all rectangles with given area.

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H640/03 Pure Mathematics and Comprehension

Question Paper

Friday 15 June 2018 – Afternoon

Time allowed: 2 hours

You must have:
- Printed Answer Booklet
- Insert

You may use:
- a scientific or graphical calculator

INSTRUCTIONS

- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- Do **not** write in the barcodes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION

- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of **20** pages. The Question Paper consists of **8** pages.

Formulae A Level Mathematics B (MEI) (H640)

Arithmetic series

$$S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n\{2a+(n-1)d\}$$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

Binomial series

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$$

where ${}^nC_r = {}_nC_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$$

Differentiation

$f(x)$	$f'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Quotient Rule $y = \dfrac{u}{v}, \dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Integration

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)(f(x))^n\,dx = \frac{1}{n+1}(f(x))^{n+1} + c$$

Integration by parts $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1-\frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$ $\quad \left(A \pm B \neq (k+\frac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0+y_n)+2(y_1+y_2+\ldots+y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x)=0$: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Probability

$\mathrm{P}(A \cup B) = \mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$ **or** $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Sample variance

$s^2 = \dfrac{1}{n-1}S_{xx}$ where $S_{xx} = \Sigma(x_i-\bar{x})^2 = \Sigma x_i^2 - \dfrac{(\Sigma x_i)^2}{n} = \Sigma x_i^2 - n\bar{x}^2$

Standard deviation, $s = \sqrt{\text{variance}}$

The binomial distribution

If $X \sim \mathrm{B}(n, p)$ then $\mathrm{P}(X=r) = {}^n\mathrm{C}_r\, p^r q^{n-r}$ where $q = 1-p$

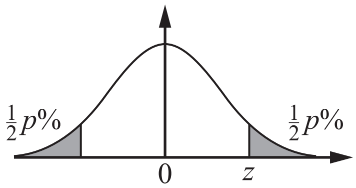
Mean of X is np

Hypothesis testing for the mean of a Normal distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\bar{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0,1)$

Percentage points of the Normal distribution

p	10	5	2	1
z	1.645	1.960	2.326	2.576



Kinematics

Motion in a straight line

$v = u+at$

$s = ut+\frac{1}{2}at^2$

$s = \frac{1}{2}(u+v)t$

$v^2 = u^2+2as$

$s = vt-\frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u}+\mathbf{a}t$

$\mathbf{s} = \mathbf{u}t+\frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = \frac{1}{2}(\mathbf{u}+\mathbf{v})t$

$\mathbf{s} = \mathbf{v}t-\frac{1}{2}\mathbf{a}t^2$

Answer **all** the questions.

Section A (60 marks)

1 Triangle ABC is shown in Fig. 1.

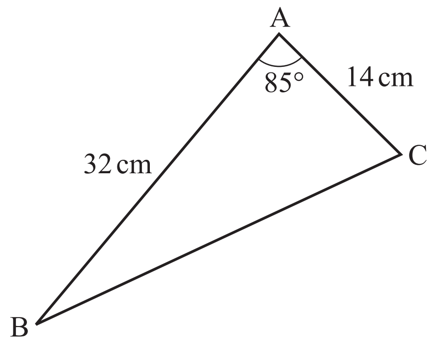


Fig. 1

Find the perimeter of triangle ABC. **[3]**

2 The curve $y = x^3 - 2x$ is translated by the vector $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$. Write down the equation of the translated curve. **[2]**

3 Fig. 3 shows a circle with centre O and radius 1 unit. Points A and B lie on the circle with angle AOB $= \theta$ radians. C lies on AO, and BC is perpendicular to AO.

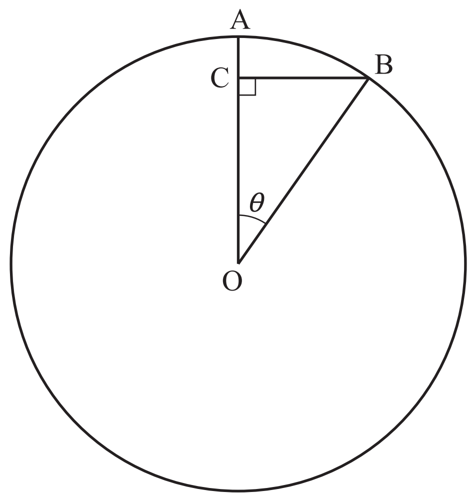


Fig. 3

Show that, when θ is small, $\mathrm{AC} \approx \frac{1}{2}\theta^2$. **[2]**

4 In this question you must show detailed reasoning.

A curve has equation $y = x - 5 + \dfrac{1}{x-2}$. The curve is shown in Fig. 4.

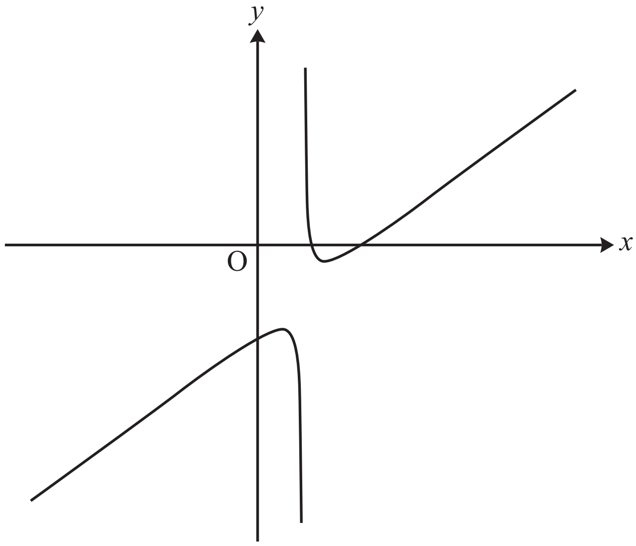


Fig. 4

(i) Determine the coordinates of the stationary points on the curve. **[5]**

(ii) Determine the nature of each stationary point. **[3]**

(iii) Write down the equation of the vertical asymptote. **[1]**

(iv) Deduce the set of values of x for which the curve is concave upwards. **[1]**

5 A social media website launched on 1 January 2017. The owners of the website report the number of users the site has at the start of each month. They believe that the relationship between the number of users, n, and the number of months after launch, t, can be modelled by $n = a \times 2^{kt}$ where a and k are constants.

(i) Show that, according to the model, the graph of $\log_{10} n$ against t is a straight line. **[2]**

(ii) Fig. 5 shows a plot of the values of t and $\log_{10} n$ for the first seven months. The point at $t = 1$ is for 1 February 2017, and so on.

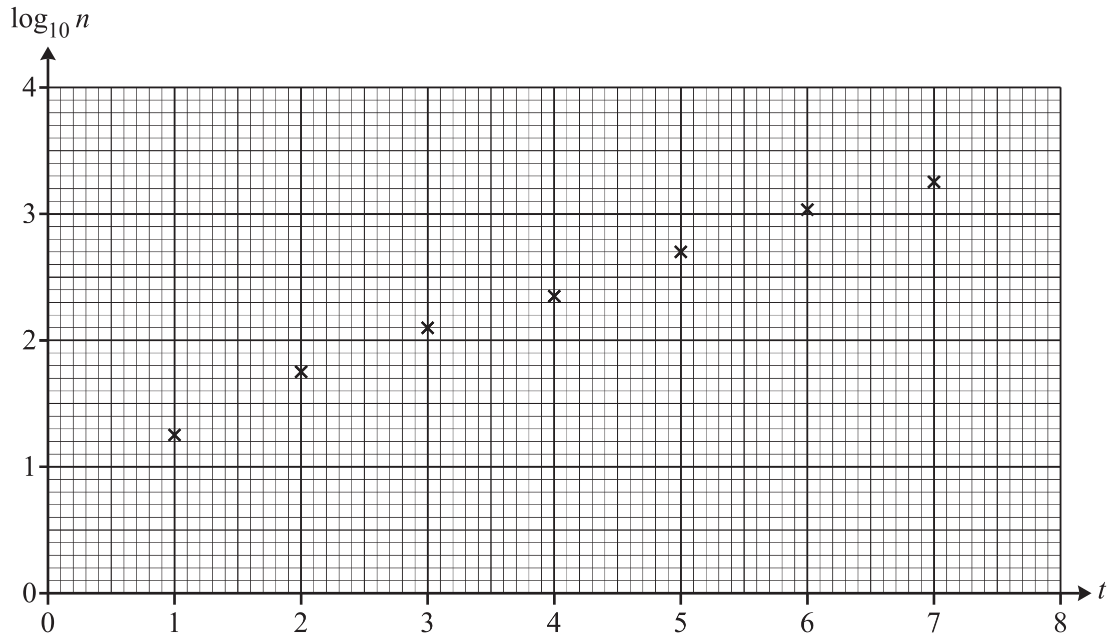


Fig. 5

Find estimates of the values of a and k. **[4]**

(iii) The owners of the website wanted to know the date on which they would report that the website had half a million users. Use the model to estimate this date. **[4]**

(iv) Give a reason why the model may not be appropriate for large values of t. **[1]**

6 Find the constant term in the expansion of $\left(x^2 + \frac{1}{x}\right)^{15}$. **[2]**

7 **In this question you must show detailed reasoning.**

Fig. 7 shows the curve $y = 5x - x^2$.

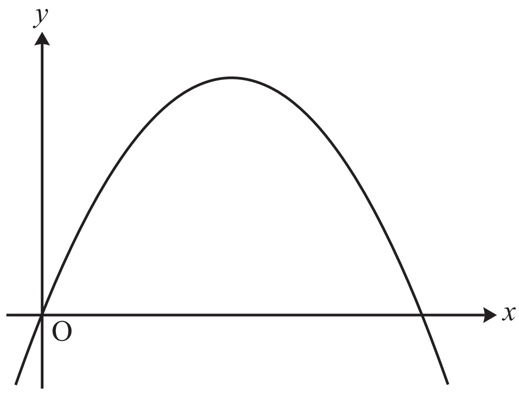


Fig. 7

The line $y = 4 - kx$ crosses the curve $y = 5x - x^2$ on the x-axis and at one other point.

Determine the coordinates of this other point. **[8]**

8 A curve has parametric equations $x = \dfrac{t}{1+t^3}$, $y = \dfrac{t^2}{1+t^3}$, where $t \neq -1$.

(i) **In this question you must show detailed reasoning.**

Determine the gradient of the curve at the point where $t = 1$. **[5]**

(ii) Verify that the cartesian equation of the curve is $x^3 + y^3 = xy$. **[3]**

9 The function $\mathrm{f}(x) = \dfrac{\mathrm{e}^x}{1-\mathrm{e}^x}$ is defined on the domain $x \in \mathbb{R}$, $x \neq 0$.

(i) Find $\mathrm{f}^{-1}(x)$. **[3]**

(ii) Write down the range of $\mathrm{f}^{-1}(x)$. **[1]**

10 Point A has position vector $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ where a and b can vary, point B has position vector $\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$ and point C has position vector $\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$. ABC is an isosceles triangle with AC = AB.

(i) Show that $a - b + 1 = 0$. **[4]**

(ii) Determine the position vector of A such that triangle ABC has minimum area. **[6]**

Answer **all** the questions.

Section B (15 marks)

The questions in this section refer to the article on the Insert. You should read the article before attempting the questions.

11 Line 8 states that $\frac{a+b}{2} \geqslant \sqrt{ab}$ for $a,\ b \geqslant 0$. Explain why the result cannot be extended to apply in each of the following cases.

(i) One of the numbers a and b is positive and the other is negative. **[1]**

(ii) Both numbers a and b are negative. **[1]**

12 Lines 5 and 6 outline the stages in a proof that $\frac{a+b}{2} \geqslant \sqrt{ab}$. Starting from $(a-b)^2 \geqslant 0$, give a detailed proof of the inequality of arithmetic and geometric means. **[3]**

13 Consider a geometric sequence in which all the terms are positive real numbers. Show that, for any three consecutive terms of this sequence, the middle one is the geometric mean of the other two. **[3]**

14 **(i)** In Fig. C1.3, angle CBD $= \theta$. Show that angle CDA is also θ, as given in line 23. **[2]**

(ii) Prove that $h = \sqrt{ab}$, as given in line 24. **[2]**

15 It is given in lines 31–32 that the square has the smallest perimeter of all rectangles with the same area. Using this fact, prove by contradiction that among rectangles of a given perimeter, $4L$, the square with side L has the largest area. **[3]**

END OF QUESTION PAPER